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An application of Gumbel's bivariate exponential distribution in estimation of warranty cost of motor cycles

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Abstract *In this article we present an application of Gumbel's bivariate exponential distribution model in the context of estimating warranty costs of motor cycles under a new warranty policy. The problem in question is as follows: Under the present two-dimensional warranty policy, repair costs (termed as warranty costs) of a motorcycle during the age of first six months or within the usage of 8,000 kilometers are borne by the company. To enhance customer satisfaction, the company wanted to bear the repair costs up to an age of one year or a usage of 12,000 kilometers. The problem is to estimate the expected hike in warranty costs if the warranty policy were revised as mentioned above. Using the past data, the problem is solved by studying the underlying renewal process. Gumbel's bivariate exponential distribution function is found to be useful in approximating the renewal function. Some practical difficulties posed by the past data in the analysis are highlighted and tackled in an interesting way.*

Introduction

A motor cycle manufacturing company was contemplating a revision in the existing warranty terms for its motorcycles (MCs). According to the present warranty terms and conditions, the company will repair any MC, through dealers, provided a complaint is reported within the age of first six months from the date of sale or within the usage of 8,000km, whichever is earlier. To enhance customer satisfaction and to be in tune with manufacturers of similar products, the company wanted to increase the warranty period to an age of one year from the date of sale or a usage of 12,000km, whichever is earlier. Consequently, the company was interested in estimating the impact of the revision of the warranty policy on the warranty costs. In this article, we present the estimate of the expected warranty cost under the proposed warranty policy.

Problem description

It has been agreed that this study be confined to 350cc MCs. Throughout this report we shall use the notation X and Y to denote the age of MC (in days) and the usage of MC (in kilometers) at the time of registering a warranty claim respectively. Typically, any two-dimensional non-renewing warranty policy (Blischke *et al.*, 1994) on automobile products is as follows: two limits x_0 and y_0

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are fixed on X and Y respectively, and whenever a warranty claim is made, it is served free of cost provided $X \leq x_0$ and $Y \leq y_0$.

We shall use the abbreviations CWP and PWP for current warranty policy and proposed warranty policy respectively. Under CWP, upon lodging a claim, a MC is repaired free of charges provided $X \leq 180$ and $Y \leq 8,000$ (i.e. $x_0 = 180$ and $y_0 = 8,000$). For PWP, $x_0 = 365$ and $y_0 = 12,000$.

The objective of this study is to estimate scientifically the expected warranty cost per MC if the current warranty policy were revised to proposed warranty policy.

Data

Background

All warranty claims are settled through dealers only. Dealers serve the warranty claims and in turn claim the warranty costs from the company. Typically, the warranty costs include the cost of replaced parts, internal and external labour costs and the local taxes as applicable.

Each time a dealer makes a warranty claim on a MC, he provides information on: engine number; dealer code; date of sale; free service details; date of failure; odometer reading; primary components repaired or replaced; consequential components replaced; and the internal and external labour costs. The company has a well-established computerized database system and every time a warranty claim is made, all the above information is stored along with the (unique) claim number.

Selection of period

For the purpose of this study we had taken data on MCs dispatched during January to March 2000 (three months). The period was chosen so as to ensure that the MCs dispatched during this period were all sold and were out of warranty period at the time of compiling the data. Particulars of the data are presented in below:

- Period: January-March 2000.
- Number of MCs dispatched: 5,044.
- Number of records (MCs that have made at least one warranty claim): 1,950.

Limitations and assumptions

It is important to note the following points about the limitations of the past data and the necessary assumptions made for modeling the problem:

- Out of the 5,044 MCs dispatched, only 1,950 had reported to the dealers – either for free maintenance service or for any complaint. There was no information on the remaining 3,094 MCs – even on the sale dates of these MCs. Since these MCs were dispatched during January-March 2000 (more than a year ago), it was assumed that all these were sold and were out of warranty (under CWP). Since it was not possible to get any information on those unreported MCs, it was assumed that those had no problems during the warranty period, and hence, their warranty costs were zero.

- It might be possible that some complaints were not brought to the dealers' notice even during the warranty period. Since it was not possible to get any information on this matter, it was once again assumed that there were no instances of this type.
- On several occasions, the company reimbursed the warranty costs even on those MCs which were out of warranty terms. This resulted in keeping only a partial information, particularly, beyond the warranty period.
- A number of obscurities were noticed when the data were scrutinized. These could have crept in either at the time of data entry or through the information provided by the dealers. It was possible to rectify some of these faulty data, others were discarded. See next section for more details.

Records

Data were extracted from the computer for the period January-March 2000 as follows. For each MC that had made at least one warranty claim, list:

- engine number;
- dealer code;
- tax rate;
- date of sale;
- free service dates; and
- odometer readings (OMRs).

For each of the claims on this MC, list:

- claim number;
- date of warranty claim and the corresponding OMR;
- codes of primary components and their quantities replaced;
- codes of consequential components replaced; and
- labour costs.

We shall call this as the record of MC. Costs of all components along with their code numbers were stored in a separate file.

Records of MCs were extracted zone-wise. However, the analysis was carried out by treating all the zones as one population. The reasons for doing this are primarily because we were interested in the overall warranty cost and also because analysis of data zone-wise was not possible as the data on sales distribution zone-wise were not available.

There were 1,950 records pertaining to this period. It was found out that 185 of these records were faulty. They had obscurities such as claim dates being prior to sale dates, abnormally high OMRs, etc. Of these records, 69 could be rectified. The remaining bad records were discarded and the production total was corrected accordingly. Thus, the total number of records that were considered for analysis purpose was 1,834. The total number of warranty claims on these records was equal to 2,086.

Censoring

The data that we have on X and Y from the records are clearly a censored one with limits on X and Y at 180 days and 8,000km respectively. There are a few records in which warranty claims have been raised with [$X > 180$, $Y < 8,000$] and [$X < 180$, $Y > 8,000$]. There could be many more MCs that have failed with $X > 180$ and/or $Y > 8,000$, but such failures have not been reported as they are out of warranty period. Thus, the information we have in the region $X > 180$ or $Y > 8,000$ is only a partial information. Subjecting such data to usual analysis will be misleading and hence, those are discarded.

Estimation of warranty cost per MC

Modeling

For any given MC, we have defined X and Y as the age and usage of the MC, in terms of number of days and OMR from the date of sale respectively, at the time of a warranty claim is made on that MC. Clearly, X and Y are multivalued functions. In fact, X and Y constitute two "renewal processes". To see this, let us consider X. Let $T_1, T_2, \ldots$ be the "between-failure times" (in terms of number of days starting from the date of sale); and let $S_n = T_1 + T_2 + \ldots + T_n$, $n \geq 1$. Then, $\{S_n, n \geq 1\}$ is a renewal process[1]. Note that whenever a claim is made, X will be equal to S_n where n is the number of claims made thus far. Similarly, let $T_1^*, T_2^*, \ldots$, be the between-failure times in terms of OMR (i.e. T_1^* is OMR at the time of first warranty claim, $T_1^* + T_2^*$ is the OMR at the time of second warranty claim and so on). Then, $\{S_n^* = \sum_{i=1}^{n} T_i^* : n \geq 1\}$ is the other renewal process Y induced by OMR.

As mentioned earlier, under any warranty policy, two limits are set on X and Y, say x_0 and y_0, so that a warranty claim is accepted provided the corresponding X, Y are such that $X \leq x_0$ and $Y \leq y_0$. Let $W_c(x, y)$ be the cost of warranty claim when $X = x$ and $Y = y$ (it is logical to think that warranty cost may depend on the age of the MC). Let $U(x, y)$ be the "bivariate renewal function" of X and Y. That is, $U(x, y)$ is the expected number of warranty claims up to $X \leq x$ and $Y \leq y$. Then the warranty cost per MC (W_h) is given by:

$$W_h = \int_0^{x_0} \int_0^{y_0} W_c(x, y) dU(x, y). \quad (1)$$

Therefore, if the distribution of W_c and the renewal function $U(x,y)$ can be approximated reasonably, then the expected warranty cost, denoted by $E(W_h)$, can be estimated under the PWP.

Warranty cost per claim (W_c)

The data on warranty costs are retrieved for each warranty claim along with the corresponding X and Y values. Examination of these data and subsequent analysis has given indications that the age and the usage of the MC do not have much impact on the warranty cost. Hence, from the angle of computations, the following simplifying assumption is made: $W_c(x,y)$ does not depend on (x,y). As an immediate consequence of this simplifying assumption, we can write from equation (1):

$$E(W_h) = E(W_c)\int_0^{x_0}\int_0^{y_0} dU(x,y) = E(W_c)U(x_0,y_0). \tag{2}$$

Ignoring the age and usage of MC, data on warranty costs are analysed for identifying the distribution of W_c. Various standard distributions are tried to fit the warranty cost data, but none of them has been a good fit (all of them have high χ^2 values). The sample mean and standard deviation of the data are 428.4 and 486.55 (see Appendix 1 for details).

Correlation between X and Y

It is evident that the variables X and Y will be correlated. A simple regression analysis is carried out regressing Y on X (see Appendix 2 for details). The least square regression line is given by $Y = 28.31X$. For $X = 180$, the predicted Y value is 5,096. This means, MCs with $Y > 5,096$ are likely to be out of warranty with respect to X. For example, a MC which had a failure on 190th day with OMR=6,000, might not have been reported for warranty claim. Consequently, the data on Y are likely to be underreported beyond 5,096km. This phenomenon is clearly evident from the analysis of data on Y.

Renewal function approximation

From equation (2), $E(W_h)$ can be estimated provided $U(x,y)$ is known. Since this function is unknown, we have to get an approximation to it using the data. It may be shown that:

$$U(x,y) = \sum_{n=1}^{\infty} \text{Prob}(S_n \leq x, S_n^* \leq y). \tag{3}$$

Computation of $\text{Prob}(S_n \leq x, S_n^* \leq y)$ requires the knowledge of the joint distribution function $F(t, t^*)$ of T_1 and T_1^* (for all i and j, it is assumed that the joint distribution function of T_i and T_j^* is F).

From the past data, we only have an empirical idea about the behavior of $U(x,y)$ in the region $G = \{(x,y) : 0 \le x \le 180, 0 \le y \le 8,000\}$. Therefore, if we can obtain a function that will reasonably match the empirical $U(x,y)$ in the region G, then that function can be used to estimate $E(W_h)$ under PWP (by taking $x_0 = 365$ and $y_0 = 12,000$).

As a first step towards finding an approximation to $U(x,y)$, we have tried to study the joint distribution function $F(t,t^*)$ of (T_1, T_1^*). In turn, we have looked at the marginal distributions of T_1 and T_1^*. A number of standard distributions have been tried to fit these marginal distributions. But none of them could fit the data on between-failure times (both on T_is and T_i^*s). As an alternative approach we have then examined the data on X.

Interestingly, it is observed that data on X exhibit the trend of an exponential function. When these data are plotted on an exponential probability graph (see Figure A2 in Appendix 3), it appear as though the data have come from an exponential distribution. This suggests that the renewal function of X can be reasonably approximated by an exponential distribution function with mean 357.0 (see Appendix 3 for details).

Marginal renewal function of Y

Prompted by the observation mentioned in the above paragraph, data on Y are also analysed in a similar fashion. However, when the entire data on Y are used, the picture has been somewhat camouflaged. Consider the exponential probability plot and the hazard rate plot drawn from the data on Y in Figure 1(a) and (b) respectively.

From Figure 1(a), we observe a drop in the probabilities in the region $\{Y > 5,000\}$. Similarly, from Figure 1(b), we observe a sudden drop in the hazard rate in the region $\{Y > 5,000\}$. We have seen from the correlation analysis of X and Y, that we do not have complete information on Y beyond the region $\{X > 180 \text{ and } Y \ge 5,096\}$. For this reason, we have used the censored data (using the criterion $X \le 180$ and $Y \le 5,096$) for studying the renewal function of Y as well as $U(X, Y)$.

The details of analysis are given in Appendix 4. From the analysis, we find that the marginal renewal function of Y can be reasonably approximated by exponential distribution with an average of 10,077.5km.

Gumbel's bivariate exponential function

It is interesting to note that both the marginal renewal functions of X and Y are reasonably approximated by exponential distribution functions. This suggests that the joint renewal function $U(x,y)$ may be approximated by a bivariate exponential distribution function. Several authors proposed different forms of bivariate exponential distribution functions (Freund, 1961; Gumbel, 1960; Marshall and Olkin, 1967). Among the various models we have tried, it is found out that Gumbel's model is the closest approximation to the data in question. Therefore, we have used Gumbel's bivariate exponential distribution function

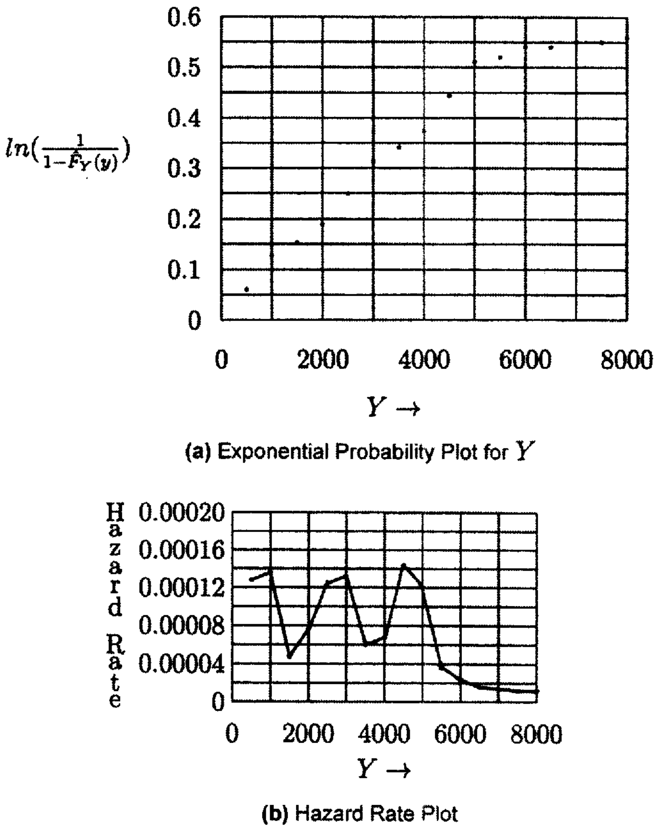

Figure 1.
Exponential probability plot for Y and hazard rate plot

to approximate $U(x, y)$. The details of the analysis are given in Appendix 5. The approximation of the renewal function of X and Y is given by:

$$U(x,y) = 1 - \exp\{-\frac{x}{\theta_x}\} - \exp\{\frac{y}{\theta_y}\} + \exp\{-\left[\left(\frac{x}{\theta_x}\right)^m + \left(\frac{y}{\theta_y}\right)^m\right]^{1/m}\},$$

where θ_x and θ_y are the parameters of the marginal renewal functions of X and Y respectively. The parameter m is estimated by the method of minimum χ^2. The estimated values of the parameters are:

$$\hat{\theta} = 357.0, \hat{\theta}_y = 10,077.5 \text{ and } \hat{m} = 357.0. \quad (4)$$

Warranty costs

To predict the warranty costs, we have used the above joint renewal function as an approximation to $U(x, y)$. Consequently, the estimate of $E(W_h)$ under PWP is given by:

$$\begin{aligned} E(W_h) &= (EW_c).U(365, 12, 000) \\ &= 428.4 \times 0.586 \\ &= 251.04 \end{aligned} \quad (5)$$

Note that we have substituted 428.4 for $E(W_c)$. This is because we are not able to fit any distribution to W_c. Since the sample size is large, we have taken the sample average as the estimate of $E(W_c)$.

No exact procedures are available for obtaining the confidence limit (upper) for the expected warranty cost per MC. However, we provide an approximate 90 percent confidence limit for $E(W_h)$ by substituting the 95 percent confidence limits for $E(W_c)$, $E(X)$ and $E(Y)$. Furthermore, this is subject to the assumption that the estimated value of m in $F(x, y)$ is exact. In this way, the approximate 90 percent upper confidence limit for $E(W_h)$ is equal to Rs.269.65.

The company has a method of estimating the warranty cost per vehicle (W_h). According to this method, $E(W_h)$ in any month is estimated as ratio of total warranty cost incurred during the past six months (including the current month) to total number of MCs dispatched during previous six months. Our estimate of $E(W_h)$ under CWP is given by:

$$\begin{aligned} E(W_h) &= (EW_c).U(180, 8, 000) \\ &= 428.4 \times 0.3714 \\ &= 159.11 \end{aligned} \quad (6)$$

It should be noted that this value is less than the average warranty cost per MC as reported by the company which is Rs.180/- for the corresponding period (computed using the method described above). Two reasons, besides the sampling errors, may be quoted for this discrepancy:

(1) the method that is used for computing W_h by the company is not exact; and

(2) keeping customer's goodwill in mind, MCs have been served beyond warranty period in a number of cases.

From the data, it is found that warranty costs has been reimbursed in a number of cases where the warranty period has expired, as reported in earlier section.

Conclusions

In this study, we have taken up the problem of estimating the expected warranty cost per motorcycle under the proposed warranty policy. This is done by analyzing the past data on warranty claims. The nature of the data imposed restrictions on subjecting the data to routine reliability analysis procedures. Therefore, the data have been subjected to censoring so as to make them amenable to meaningful analysis. The problem has been tackled by analyzing the underlying bivariate renewal process. Exploiting the special feature of the data (that the marginal renewal functions are approximated by exponential distribution functions), the bivariate renewal function is fitted with a bivariate exponential distribution function. Using this fit for the renewal function, the expected warranty cost per motorcycle is estimated.

The estimated expected warranty cost per MC under the proposed warranty period of 365 days or 12,000km, whichever is earlier, is equal to Rs.251.04 and the approximate 90 percent upper confidence limit for the same is Rs.269.65/-.

Notes

1. It is assumed that T_is are independent random variables with a common distribution function; it is also assumed that the repairs are done instantaneously.
2. The graph shown here is plotting of $\ln(1/\hat{R}(x))$ against failure time which is equivalent to plotting the data on an exponential probability graph.

References

Freund, R.J. (1961), "A bivariate extension of the exponential distribution", *Journal of the American Statistical Association*, Vol. 56, pp. 971-7.

Gumbel, E.J. (1960), "Bivariate exponential distributions", *Journal of the American Statistical Association*, Vol. 55, pp. 698-707.

Marshall, A.W. and Olkin, I. (1967), "A multivariate exponential distribution", *Journal of the American Statistical Association*, Vol. 62, pp. 30-44.

Further reading

Blischke, W.R. and Murthy, D.N.P. (1994), *Warranty Cost Analysis*, Marcel Dekker, New York, NY.

Johnson, N.L. and Kotz, S. (1972), *Distributions in Statistics*, John Wiley & Sons, New York, NY.

Nelson, W. (1982), *Applied Life Data Analysis*, John Wiley & Sons, New York, NY.

Appendix 1. Distribution of warranty cost

The database maintains a file in which component costs are stored. From the records, each claim has the number of primary components replaced and/or repaired, the consequential components replaced (consequential components are always replaced) and the labour costs (both internal and external). For each claim, warranty cost is worked out using the above information. This includes the taxes on net dealer prices. The data are summarized in the form of a frequency table and shown in Table AI.

We tried to fit distributions such as exponential, lognormal and Weibull. But none of these could fit well. The sample mean and standard deviation, as computed from the data, are 428.4 and 486.55. Since the sample size is large (= 2, 086), the upper 95 percent confidence limit for $E(W_c)$ is obtained using normality assumption (taking 428.4 as the estimate for mean). This is equal to 449.26.

Appendix 2. Correlation between X and Y

As the number of days (X) increase, naturally the distance covered by the vehicle, OMR (Y) also increase. Figure A1 shows the scatter diagram between X and Y. A simple regression analysis

Warranty cost	Freq.	Cum. freq.
0 to 200	871	871
200 to 400	522	1,393
400 to 600	253	1,646
600 to 800	147	1,793
800 to 1,000	81	1,874
1,000 to 1,200	54	1,928
1,200 to 1,400	21	1,949
1,400 to 1,600	41	1,990
1,600 to 1,800	28	2,018
1,800 to 2,000	23	2,041
2,000 to 2,200	16	2,057
2,200 to 2,400	7	2,064
2,400 to 2,600	5	2,069
above 2600	17	2,086

Table AI.
Warranty cost per claim

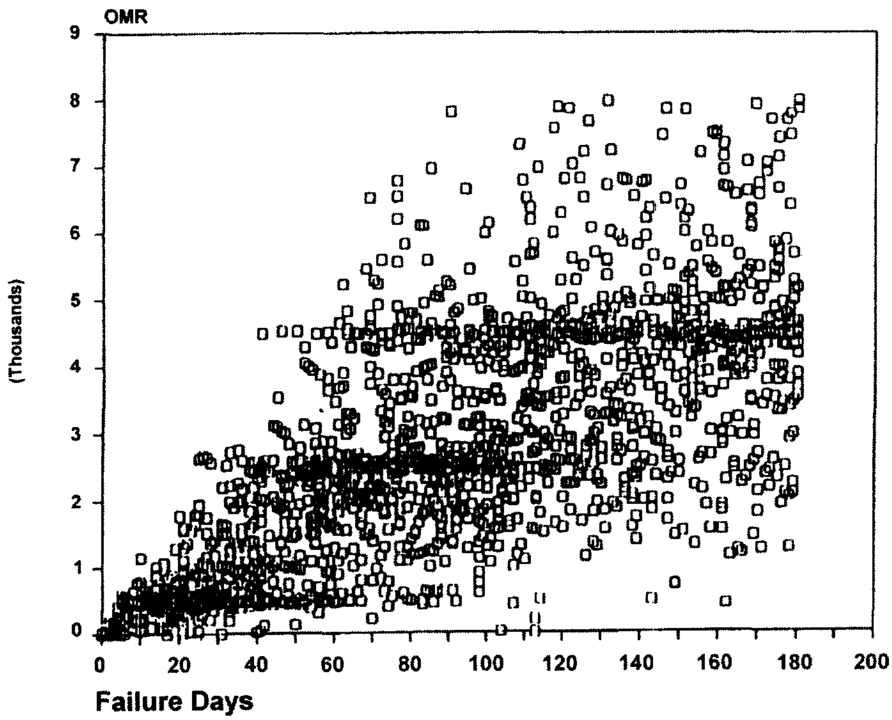

Figure A1.
Scatter diagram

(with zero intercept) is carried out regressing Y on X. The following are the results of the analysis. The fitted line is:

$$Y = 28.31X.$$

For $x = 180$, the y-value is equal to 5,096.

Appendix 3. Marginal renewal function of X

We have defined X as the age of MC, in terms of number of days, at the time of making a claim. It has been mentioned earlier that X corresponds to a renewal process $\{S_n = \sum_{i=1}^{\infty} T_i : n \geq 1\}$, where T_is are the between-failure times. For any time interval $[a, b]$, the number of observations on X that lie in this interval is the number of claims that have been made during this time period $[a, b]$. If $U_X(t)$ is the expected number of claims made during the time interval $[0, t]$, then:

$$U_X(t) = \sum_{n=1}^{\infty} \text{Prob}(S_n \leq t).$$

Study of $U_X(t)$ requires the knowledge of distribution of T_is. A number of standard distributions are tried to fit the data on between-failure times. But the data do not fit into any of these models. However, when the data on X have been examined, those appear as though they have come from an exponential distribution. Consequently, we have adopted the following heuristic approach to approximate $U_X(t)$.

The censored data on X (using the condition $X \leq 180$) are tabulated in the form of a frequency table (see Table AII). The ratios of these frequencies to total number of MCs dispatched (f_i/N) give us an empirical approximation to the renewal density function $u_X(x)$, i.e.:

$$u_X(X) = \frac{dU_X(x)}{dx}.$$

If we treat these data as failure times out of N items put on test (dispatched), then we can test whether these data come from an exponential distribution. Taking N as the total number of vehicles under consideration, the data are plotted on an exponential probability graph (see Figure A2[2]). The graph suggests that the renewal function can be approximated by exponential

Table AII. Analysis of data on X

Failure time	Freq.	Cum. freq.	$\hat{F}_X(x)$	$1 - \hat{F}_X(x)$	$\ln(1/(1 - \hat{F}_X(x)))$
0 to 15	186	186	0.0384	0.9616	0.0392
15 to 30	240	426	0.0862	0.9138	0.0901
30 to 45	186	612	0.1238	0.8762	0.1322
45 to 60	157	769	0.1556	0.8444	0.1691
60 to 75	183	952	0.1926	0.8074	0.2139
75 to 90	195	1,147	0.2320	0.7680	0.2640
90 to 105	169	1,316	0.2662	0.7338	0.3096
105 to 120	142	1,458	0.2950	0.7050	0.3495
120 to 135	115	1,573	0.3182	0.6818	0.3831
135 to 150	104	1,677	0.3393	0.6607	0.4144
150 to 165	107	1,784	0.3609	0.6391	0.4471
165 to 180	125	1,909	0.3862	0.6138	0.4881

Note: After censoring, the total number of claims made is equal to 1,909

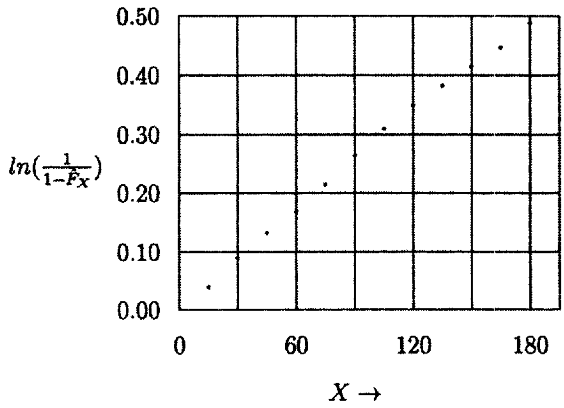

Figure A2. Exponential probability plotting for X

distribution function. However, when we test for goodness of fit, the resulting χ^2_{11} value is found to be 40.3. The estimated mean of the distribution is equal to 357.0. Hence, we have taken:

$$U_X(t) = 1 - \exp(-t/357), \quad t \geq 0,$$

as our approximation to $U_X(t)$.

Appendix 4. Marginal renewal function of *Y*

While using the entire data to study the renewal function of Y, an interesting picture has been observed. There is a sudden drop in the hazard rate in the region $\{Y > 5,000\}$ which could be due to lack of complete information on Y beyond the region $\{X > 180$ and $Y \geq 5,096\}$. Hence, the data are censored by using the criterion $\{X \leq 180$ and $Y \leq 5,096\}$ for studying the renewal function of Y.

As in the case of X, we have carried out a similar analysis for Y. The data are shown in Table AIII. The exponential probability plot (see Figure 1(a)) suggests that the marginal renewal function of Y, $U_Y(y)$, can be approximated by an exponential distribution function. Even here, the χ^2_9-value for the goodness of fit is 28.0. Nonetheless, in the light of the probability plot, we

Failure odometer	Freq.	Cum. freq.	$F(y)$	$R(y)$	$h(y)$	ln $(1/R(y))$
0 to 500	285	285	0.0589	0.9411	0.00013	0.0607
500 to 1,000	294	579	0.1196	0.8804	0.00014	0.1274
1,000 to 1,500	110	689	0.1423	0.8577	0.00005	0.1535
1,500 to 2,000	149	838	0.1731	0.8269	0.00007	0.1901
2,000 to 2,500	231	1,069	0.2208	0.7792	0.00012	0.2495
2,500 to 3,000	232	1,301	0.2687	0.7313	0.00013	0.3130
3,000 to 3,500	100	1,401	0.2894	0.7106	0.00006	0.3417
3,500 to 4,000	112	1,513	0.3125	0.6875	0.00007	0.3747
4,000 to 4,500	224	1,737	0.3588	0.6412	0.00014	0.4444
4,500 to 5,100	172	1,909	0.3943	0.6057	0.00012	0.5014

Table AIII. Analysis of data on Y

have approximated $U_Y(y)$ by exponential distribution function with mean equal to 10,077.5. Thus, our approximation for $U_Y(y)$ is given by:

$$U_Y(y) = 1 - \exp(-y/10,077.5), \quad y \geq 0.$$

Appendix 5. Joint renewal function of X and Y

A bivariate distribution function whose marginals are exponential is called a bivariate exponential distribution (BED). Several models have been suggested by different authors for the BED (Freund, 1961; Gumbel, 1960; Marshall and Olkin, 1967). Among various models tried, it has been observed that the following Gumble's model is the closest approximation to the data in question:

$$F(x,y) = 1 - \exp\{-\frac{x}{\theta_x}\} - \exp\{-\frac{y}{\theta_y}\} + \exp\{-\left[\left(\frac{x}{\theta_x}\right)^m + \left(\frac{y}{\theta_y}\right)^m\right]^{1/m}\}.$$

Here m is the bivariate parameter.

The following method is adopted in selecting the model for BED: the data on (X, Y) are first classified into a two-way table and the observed cell frequencies are determined from the data (see Table AIV). Then, using a BED model the expected cell frequencies are worked out. This calculation, however, requires the parametric values of the BED model.

For example, in the Gumbel's model above, we need values of θ_x, θ_y and m to compute the expected cell frequencies. In this case, we have used $E(X)$ and $E(Y)$ for θ_x and θ_y respectively, and then we have found out that value of m which minimizes the sum of squares of the deviations between the observed and expected frequencies. Finally, the BED model with closest expected frequencies to the observed ones is chosen as the best approximation model for the data in question. This has turned out to be Gumbel's model given below, with the parameters replaced by their estimates:

$$F(x,y) = 1 - e^{-x/357.0} - e^{-y/10,077.5} + e^{-[(x/357.0)^{3.15} + (y/10,077.5)^{3.15}]^{1/3.15}},$$

The expected cell frequencies according to this model are given in Table AV.

The expected warranty cost per MC is equal to:

$$E(W_h) = \{ \begin{array}{l} E(Wc).F(180, 8,000) \text{ under CWP} \\ E(Wc).F(365, 12,000) \text{ under PWP}' \end{array} \quad \text{(A1)}$$

$$E(W_h) = \{ \begin{array}{l} 159 \text{ under CWP} \\ 251.04 \text{ under PWP} \end{array} \quad \text{(A2)}$$

No exact methods are available for obtaining the confidence interval on $E(W_h)$. The approximate 90 percent upper confidence limit on $E(W_h)$ is obtained by independently substituting the approximate 95 percent upper confidence limit for $E(W_c)$ and the approximate lower confidence limits for θ_x and θ_y. Furthermore, the value of m is assumed to be exact in the computation of confidence limits. The upper confidence limit for $E(W_h)$ under PWP is equal to Rs.269.65.

Table AIV. Two-way table for observed joint distribution of *X* and *Y*

Day (x)	OMR (y) 0 to 500	500 to 1,000	1,000 to 1,500	1,500 to 2,000	2,000 to 2,500	2,500 to 3,000	3,000 to 3,500	3,500 to 4,000	4,000 to 4,500	4,500 to 5,100
0 to 15	128	56	2	0	0	0	0	0	0	0
15 to 30	88	124	12	12	0	4	0	0	0	0
30 to 45	38	64	28	19	21	11	2	1	2	0
45 to 60	15	21	16	28	34	26	5	4	5	3
60 to 75	5	14	10	19	45	40	14	10	17	9
75 to 90	4	8	16	16	39	46	11	13	20	22
90 to 105	2	4	10	22	30	40	13	11	21	16
105 to 120	3	2	3	10	20	17	15	19	33	20
120 to 135	0	0	4	8	9	20	15	13	24	22
135 to 150	1	1	2	7	16	12	9	10	21	25
150 to 165	1	0	3	6	5	8	8	16	36	24
165 to 180	0	0	4	2	12	8	8	15	45	31

Table AV. Two-way table for expected joint distribution of *X* and *Y*

Day (*x*)	OMR (*y*) 0 to 500	500 to 1,000	1,000 to 1,500	1,500 to 2,000	2,000 to 2,500	2,500 to 3,000	3,000 to 3,500	3,500 to 4,000	4,000 to 4,500	4,500 to 5,100
0 to 15	166	25	5	2	1	0	0	0	0	0
15 to 30	54	82	31	11	5	3	2	1	1	0
30 to 45	13	58	50	27	14	7	4	3	2	1
45 to 60	5	29	44	35	22	13	8	5	4	2
60 to 75	2	14	30	33	27	19	12	8	6	4
75 to 90	1	8	19	26	26	21	16	11	8	6
90 to 105	1	5	12	19	22	21	17	13	10	8
105 to 120	0	3	8	14	18	19	17	14	12	9
120 to 135	0	2	5	10	14	16	16	14	12	10
135 to 150	0	1	4	7	10	13	14	14	12	11
150 to 165	0	1	3	5	8	10	12	12	12	11
165 to 180	0	1	2	4	6	8	10	11	11	10